Optimal Replacement without Knowledge of the Hazard Function

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Abstract

Relying on few assumptions, I present a simple model which assists the decision of whether to replace a degrading system now, or wait. While most optimal replacement-repair models provide guidance on the event, condition, or age of a system which dictates an optimal (usually expectation) replacement time, they all rely on knowledge about the hazard function. In our case, the available models required more knowledge about the hazard function than we could determine, and our decision process requires simplicity. However, even with almost no knowledge about the degradation behaviour, or severely limiting assumptions, we are able to provide decision makers with metrics which add assurance that a system should be replaced. It can also be used to prioritize systems for replacement in cases of limited budget, even when we only know costs, age, and current rate of occurrence of failures.

1. Background

1.1 Brief Literature Reference

There is a wealth of literature fitting under the general subject of optimal repair-replacement modelling. For excellent, yet dated, breakdowns of the literature in this subject area, see (Valdez-Florez & Feldman, 1989), (Pierskalla & Voelker, 1976). Generally speaking, the literature on this subject assumes some functional characteristic about the degradation, or rather the increase in maintenance cost over time. They assume that the system or component has a hazard rate whose function is known, and usually it is strictly non-decreasing, except where they show there is no optimal replacement point.

While each paper is an important extension of the state of the art, it is my belief that these assumptions are often motivated by easy math, and the need to publish; not by a particular application. Mostly, the papers published as of late fit into one of these categories:

1. extensions into more restrictive assumptions which improve optimality, but require more information about the component or system; or
2. expansions of condition versus event versus age replacement, or some mix, increasing the complexity of the decision or maintenance processes.

For an example of the former case, I point to (Deshpande & Singh, 1995). For an example of the latter case, see (Bagai & Jain, 1994). However, while these works and the many others not sited are all important, my experience has shown that they are not easy to apply because they require more information, or are complex. Where there exist the largest opportunities for maintenance optimization, unfortunately analyst resources are lacking or missing, and the important decisions are made by engineers who define, and therefore must comprehend fully, the decision process.

1.2 Our motivation

As I've already indicated, our situation does not allow easy application of the existing models. Traditional condition- and event-based replacements do not apply well to our case because our system is complex, repairs are inexpensive relative to replacement, information on repairs is kept for little more than one *cycle*, and the dominating degradation is slow; it is difficult to know when the right event or condition has taken place. Age replacement applies to our case better, but not well enough. A strictly age-based replacement policy does not apply well because of the high degree of variation in degradation, and therefore repairs. In addition, replacement decisions are made by looking at statistics on a very large set of systems, and engineers search for opportunities to spend predetermined budget. Conditions, events, and age are fuzzy at best. And generally, the available models do not help because we do not know enough about the hazard function being generated by degradation.

Instead, I've had to develop a simplified approach, solve for the age of optimal replacement under a certain set of assumptions, then invert to relax assumptions. This approach allows us to take into account what we do know about a particular system being investigated, even when that information is minimal. The end result is a decision support metric based on optimal replacement age, as defined by a degradation condition. While age is initially fuzzy, it can be determined with sufficient certainty.

1.3 Nomenclature

C	Cost per unit of time, or time average cost
C_1	Cost of a minimal repair
C_2	Cost of a replacement
M	Change over time, or slope, of the rate of occurrence of failures, over time
T	Age of the system at time of study
T_1	Age at which degradation begins in the system
T_2	Age at which the system is replaced
T_2^*	Optimal age for system replacement
T_{1C}	Given the system is at the optimal age for replacement, this is the minimum T_1 such that the system should have been replaced.
Y_b	Rate of occurrence of failures which are basic, not due to degradation failure modes
Y_d	Rate of occurrence of failures due to degradation
P	probability that the optimal replacement time has occurred by the age of the system, or the proportion of the timeline (0, T) over which T_1 would dictate optimal replacement has occurred.
cycle	The period of time over which repair patterns repeat. In our application, this is 12 months, and represents that seasonality effects on repair, particularly due to the degradation failure mode.

2. Model

In this application, we have very few assumptions we can validly make. Here is what we know:

1. The statistical-average cost of a minimal repair C_1, and the expert-anticipated (not a statistical-average) cost of replacement C_2 .
2. The age of the system or subsystem being considered for replacement T .
3. The rate of occurrence of failures for the last *cycle* of time, but separated into degradation Y_d , and non-degradation Y_b .

The first two items we find through investigation, inspection, and from basic business metrics. We can determine the occurrence rate of each failure category for a particular system through repair records over the last *cycle*. Due to the cyclical effects on failure, one *cycle* of data is not sufficient for developing the hazard functions. So we must make one key assumption:

> Assumption – At repair, the system or subsystem is inspected such that the cause of failure is categorized as due to degradation, or an alternate cause; and this information is reliable. Therefore, if degradation has occurred over the last *cycle*, we know it began some time in (0, T). Once it has occurred, degradation will continue until replaced. Without any additional knowledge, our minimal assumption is that the degradation is linearly increasing in time once it begins.

This leads us to the following process:

- We take the current *cycle* of repair information, and determine both Y_d & Y_b.
- We use this information, with the age T, and costs C_1 & C_2, to determine the optimal replacement age T_2^* conditional on the assumed beginning degradation time T_1, $T_2^* \mid T_1$.
- Then we take available information on the likelihood of each possible value of T_1 to determine the probability that the optimal replacement time has occurred for this system or subsystem, P.

The rest of this paper outlines the equations used to follow this process.

2.1 Optimal replacement age

For a given M, the total cost per unit time is

$C = \dfrac{C_1 Y_b T_2 + C_1 M\, 0.5(T_2 - T_1)^2 + C_2}{T_2}$. The first derivative with respect to the age, and the first order necessary condition, is then

$\dfrac{\partial C}{\partial T_2} = 0.5 C_1 M - (C_1 M\, 0.5 T_1^2 + C_2) T_2^{-2} = 0$; and the second order sufficient condition is

$\dfrac{\partial^2 C}{\partial T_2^2} = (C_1 M T_1^2 + 2C_2) T_2^{-3} > 0$, which is concave upward, and therefore provides a minimum cost at

$T_2^* = \sqrt{\dfrac{C_1 M\, 0.5 T_1^2 + C_2}{C_1 M\, 0.5}}$.

2.2 Unknown rate of change in degradation repair over time

As we do not actually know the rate of change in degradation, we estimate it with M, defined to be

$M = \dfrac{Y_d}{T - T_1}$. Therefore, $T_2^* \mid T_1 = \sqrt{\dfrac{0.5 C_1 Y_d T_1^2 - C_2 T_1 + C_2 T}{0.5 C_1 Y_d}}$ The roots of this equation are given by $\dfrac{C_2 \pm \sqrt{C_2^2 - 2C_1 Y_d (C_2 T - C_1 Y_d\, 0.5 T_2^{*2})}}{C_1 Y_d}$. At this point, we need to demonstrate the conditions in which one root is in our range of interest, and the replacement decisions which result. That exercise is for future work, as it would not fit within the limits of this extended abstract.

2.3 Likelihood of Valid Replacement

However, it is simple and effective to place the equation for T_2^* into a spreadsheet, and show the values of T_1, with $0 < T_1 < T$, for which $T_2^* < T$ for any particular case. By taking a fine enough granularity for T_1, we can approximate the value of P by adding up the number of times $T_2^* \mid T_1 \leq T$, and dividing by the total number of events. $P = \dfrac{T - (T_{1C})}{T}$.

3. Conclusions

Even though we do not know the hazard function, we know enough about the degradation mechanisms of our system to say that once degradation had begun, it will continue. If we make the simplest assumption that the degradation is linear in nature, we can define the optimal age of replacement conditional on the age at which the degradation begins. We can further use this result to relax the condition, and find the likelihood that the optimal replacement time has come. Knowing that the actual optimal condition is sensitive to the hazard function, our results are still useful as our metric P.

In this last step, we make an assumption yet undocumented: the likelihood that the degradation began at T_1 is uniform. Realistically, with the knowledge given in (Rupe, 1999), we could use the distribution of T_1 to determine P, and improving this metric.

Another yet undocumented assumption common to almost all replacement optimization modelling exercises including this one is that a replacement would behave the same as the original. In reality, the expected behaviour of the replacing equipment may be better or worse depending on many factors. In that case, we must adapt these models much in the way technology replacement is considered in (Rupe, 2000). Until that is considered, the metric P as defined here is not useful.

The metric P becomes a comparison risk estimate for system replacement, useful by engineers who have a limited budget, and must make the best financial decision for the company.

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